Journal of Algebra 368 (2012) 231–236

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A new characteristic subgroup of a p-stable group

George Glauberman [a,*], Ronald Solomon [b]

[a] University of Chicago, Chicago, IL 60637, United States
[b] Ohio State University, Columbus, OH 43210, United States

ARTICLE INFO

Article history:
Received 22 January 2012
Available online 20 July 2012
Communicated by Aner Shalev

Dedicated to our friend, Said Sidki, on the occasion of his 70th birthday

Keywords:
Sylow p-subgroup
p-Stable group
Characteristic subgroup

ABSTRACT

We define a new non-identity characteristic subgroup, $D^*(S)$, of a finite p-group S, and prove that $D^*(S)$ is characteristic in a significant family of p-stable finite groups G having S as a Sylow p-subgroup. This result is analogous to an earlier result about $ZJ(S)$ by the first author and has a very short proof.

1. Introduction

In 1968, the first author showed [GG], for all odd primes p, that $ZJ(S)$ is a characteristic subgroup in a significant family of p-stable groups G of which S is a Sylow p-subgroup. This result was applied in several major subtheorems of the classification of finite simple groups, in particular the classification of groups with abelian Sylow 2-subgroups [W], for then the p-stability hypothesis could be established easily for many proper subgroups. (Note that, by [HB, pp. 492–493], a p-solvable group G is p-stable whenever either $p \geqslant 5$, or $p = 3$ and G has abelian Sylow 2-subgroups.)

In this paper we will introduce two new characteristic subgroups, $D^*(S)$ and $D^*_e(S)$, of a finite p-group S. We denote by $O_p(G)$ the largest normal p-subgroup of the finite group G, and we prove that both $D^*(S)$ and $D^*_e(S)$ are characteristic in any p-stable group G in which $C_G(O_p(G)) \leqslant O_p(G)$ and S is a Sylow p-subgroup.

***D^**-Theorem.** *Let p be a prime and G a non-identity p-stable finite group with $C_G(O_p(G)) \leqslant O_p(G)$. If S is a Sylow p-subgroup of G, then $D^*(S)$ and $D^*_e(S)$ are non-identity characteristic subgroups of G.*

* Corresponding author.
E-mail addresses: gg@math.uchicago.edu (G. Glauberman), solomon@math.ohio-state.edu (R. Solomon).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.07.008

These subgroups have a useful property (Lemma 1(b) below) which is not satisfied by $ZJ(S)$, and which permits a very short proof of the theorem above. We do not require that p be an odd prime, but the notion of p-stability (defined below) is not of interest when $p=2$. On the other hand, in [St], Bernd Stellmacher proved an analogue of [GG] for $p=2$ and a subgroup of $\Omega_1(ZJ_e(S))$. This raises the question of whether an analogue of Stellmacher's theorem can be proved for a characteristic subgroup similar to $D_e^*(S)$.

We thank Avinoam Mann for stimulating conversations and suggestions that led to this article. In particular, the definitions of $D^*(S)$ and $D_e^*(S)$, the first proof of Theorem 1, and the proofs of Proposition 1 and Theorem 2, are modeled after the definition of $D(S)$ and the corresponding proofs in his paper [M].

During the preparation of this paper, the first author enjoyed the hospitality of the Universities of L'Aquila and Rome, and helpful comments from N. Gavioli, V. Monti, V. Naik, and C. Scoppola. The second author has benefited from the financial support of the Mathematical Research Institute of the Ohio State University for his research. We thank these individuals and institutions warmly.

2. Definitions and proofs

First we define the subgroups $D^*(S)$ and $D_e^*(S)$.

Definition 1. Let S be a finite p-group. We let $\mathcal{D}(S)$ (resp., $\mathcal{D}_e(S)$) denote the set of all abelian (resp., elementary abelian) subgroups A of S satisfying

$$\text{If } x \in S \text{ and } \langle A, x\rangle \text{ has nilpotence class at most 2, then } x \text{ centralizes } A. \tag{$*$}$$

Also, set $D^*(S) := \langle A\colon\ A \in \mathcal{D}(S)\rangle$, and define $D_e^*(S)$ analogously.

The following two observations are immediate from the definitions. We write $D_{(e)}^*(S)$ to denote either $D^*(S)$ or $D_e^*(S)$.

Lemma 1.

(a) *$Z(S) \leqslant D^*(S)$ and $\Omega_1(Z(S)) \leqslant D_e^*(S) \leqslant D^*(S)$; and*
(b) *if $D_{(e)}^*(S) \leqslant T \leqslant S$, then $D_{(e)}^*(S) \leqslant D_{(e)}^*(T)$.*

We shall also use the following observation which relates condition $(*)$ to p-stability as defined below.

Lemma 2. *Suppose A is abelian and x normalizes A. Then $\langle A, x\rangle$ has nilpotence class at most 2 if and only if $[A, x, x] = 1$.*

Proof. Clearly it suffices to show that if $[A, x, x] = 1$, then $B := \langle A, x\rangle$ has class at most 2. Now $[A, x]$ is contained in A and centralized by x, hence is contained in $Z(B)$. Therefore, $B/Z(B)$ is abelian, and B has class at most 2. □

Our first main result concerning these subgroups is the following theorem, for which we offer two proofs. In both proofs, we let $\mathcal{D}$ denote either $\mathcal{D}(S)$ or $\mathcal{D}_e(S)$, and let D denote the join of all members of $\mathcal{D}$. Here and later, we use notation such as $Z(A \bmod B)$, when $B \lhd A$, to denote the full pre-image in A of $Z(A/B)$.

Theorem 1. *$D_{(e)}^*(S)$ is the unique maximal member of $\mathcal{D}_{(e)}(S)$. In particular, if $S \neq 1$, then $D_{(e)}^*(S)$ is a non-identity characteristic (elementary) abelian subgroup of S.*

Proof. Let $Y = Z(D)$ and $X = Z(D \bmod Y)$, so that $X \geqslant Y$ and $X/Y = Z(D/Y)$. If $x \in X$ and $A \in \mathcal{D}$, then $[A, x] \leqslant Y$. Therefore, $\langle A, x\rangle$ has class at most 2. Hence, by $(*)$, $[A, x] = 1$ for all $A \in \mathcal{D}$. Thus $[D, x] = 1$ for all $x \in X$. It follows that $X = Y$ and $Z(D/Y) = 1$, whence $D/Y = 1$. Therefore, $D = Y = Z(D)$, and D is abelian. Moreover, if $\mathcal{D} = \mathcal{D}_e(S)$, then D is the join of elementary abelian subgroups, and hence elementary abelian.

Now let $x \in S$ with $\langle D, x\rangle$ of class at most 2. Then $\langle A, x\rangle$ has class at most 2 for all $A \in \mathcal{D}$. So $[A, x] = 1$ for all $A \in \mathcal{D}$. Again, we conclude that $[D, x] = 1$. Hence $D \in \mathcal{D}$, as claimed. $\square$

The proof above was suggested by that of Theorem 7(a) in [M], which, in turn, was inspired by the proof of Proposition 3 in [Is]. For variety we present a second proof of Theorem 1. This has the advantage that it can easily be adapted to some other characteristic subgroups defined by variations on the condition $(*)$. Here, we proceed in a short sequence of easy lemmas.

Lemma 3. *Let A and B be members of $\mathcal{D}$ that normalize each other. Then $AB \in \mathcal{D}$.*

Proof. As $[A, B] \leqslant A \cap B \leqslant Z(AB)$, we see that $[A, x, x] = 1$ for all $x \in B$, whence $[A, x] = 1$ for all $x \in B$. Hence AB is abelian, and elementary abelian if both A and B are. If $x \in S$ with $\langle AB, x\rangle$ of class at most 2, then $\langle A, x\rangle$ and $\langle B, x\rangle$ are likewise of class at most 2, whence $[A, x] = [B, x] = 1$ and so $[AB, x] = 1$. Thus $AB \in \mathcal{D}$. $\square$

Lemma 4. *Let M be a maximal member of $\mathcal{D}$. Then $M \lhd S$.*

Proof. Suppose not and let $T = N_S(M) < S$. Choose x in $N_S(T) - T$. Then M and M^x are distinct normal subgroups of T. As both are in $\mathcal{D}$, Lemma 3 implies that $M < MM^x \in \mathcal{D}$, a contradiction. $\square$

Second proof of Theorem 1. Let M and M_1 be two maximal members of $\mathcal{D}$. By Lemma 4, both are normal in S. Then, by Lemma 3, $MM_1 \in \mathcal{D}$, whence $M = MM_1 = M_1$. Thus M contains every member of $\mathcal{D}$, whence $M = \langle A\colon\ A \in \mathcal{D}_e(S)\rangle = D$ is the unique maximal member of $\mathcal{D}$, as claimed. $\square$

We recall the definition of the Thompson subgroup, $J(S)$. Let $\mathcal{A}(S)$ denote the set of abelian subgroups of S of maximum order. Then $J(S) := \langle A\colon\ A \in \mathcal{A}(S)\rangle$. We write $ZJ(S)$ for the center of $J(S)$. Note that the definitions of $J(S)$ and $ZJ(S)$ require "global" information about S, namely, all abelian subgroups of maximum order in S. In contrast, the definitions of $(*)$, $D^*(S)$, and $D_e^*(S)$ are "local", like Mann's definition of $D(S)$. This explains why Lemma 1(b) is valid for $D^*(S)$ and $D_e^*(S)$, while the analogue for $ZJ(S)$ is not valid, as we shall illustrate at the end of the paper with a family of examples.

Later, we will show that $Z(S) \leqslant D^*(S) \leqslant ZJ(S)$. We note here some examples to show that neither equality need hold. However, when $p = 2$, we observe the following equality.

Lemma 5. *Let S be a finite 2-group. Then $D_e^*(S) = \Omega_1(Z(S))$.*

Proof. Let $D = D_e^*(S)$ and suppose that $D > \Omega_1(Z(S))$. Choose $x \in S - C_S(D)$ of minimum order. Then $x^2 \in C_S(D)$ and so $[D, x, x] = 1$. But then, by $(*)$ and Lemma 2, x centralizes D, contrary to the choice of x. $\square$

Examples. Note that if S has class at most 2, then $D^*(S) = Z(S)$ and $D_e^*(S) = \Omega_1(Z(S))$.

In particular, let $S_1 = \langle A, x\rangle$, where A is a normal elementary abelian subgroup of S_1, $[A, x, x] = 1$, $x^p = 1$, and $|[A, x]| \geqslant p^2$. (Thus x, as a linear operator on A, has minimum polynomial $(t-1)^2$ and has at least two Jordan blocks of size 2.) Then $D_e^*(S_1) = D^*(S_1) = Z(S_1) < ZJ(S_1) = J(S_1) = A$.

On the other hand, for p odd, let $S_2 = \langle A, x\rangle$, with A normal elementary abelian and $x^p = 1$, but $[A, x, x] \neq 1$. (Thus, $|A| \geqslant p^3$, and x, as a linear operator on A, has minimum polynomial $(t-1)^k$ with $k \geqslant 3$.) Then $Z(S_2) = C_A(x) < A = D_e^*(S_2) = D^*(S_2) = J(S_2) = ZJ(S_2)$. If $p = 2$, then $S_2 = D_{16}$

affords an example where $Z(S_2) < D^*(S_2)$, since $D^*(S_2) = J(S_2)$, the maximal cyclic subgroup of S_2.

Finally, if $S = S_1 \times S_2$, then $Z(S) < D^*(S) < ZJ(S)$, and, if p is odd, then $D_e^*(S) = D^*(S)$.

We now proceed to the proof of D^*-Theorem. We give the definition of a p-stable group G from [HB, p. 492].

Definition 2. Let p be a prime. A finite group G is p-stable if whenever P is a p-subgroup of G, and $g \in N_G(P)$ with $[P, g, g] = 1$, then g lies in $O_p(N_G(P) \bmod C_G(P))$, the full pre-image of $O_p(N_G(P)/C_G(P))$ in $N_G(P)$.

Proof of D^*-Theorem. Again, let $D := D^*(S)$ (or $D_e^*(S)$). If $D \lhd G$, then $D = D^*(S_1)$ for every Sylow p-subgroup S_1 of G, whence D is a characteristic subgroup of G. Thus, it will suffice to prove that D is normal in G.

Let $T = O_p(G)$. Then as $T \leqslant S$ and $D \lhd S$, we have $[T, D, D] \leqslant [D, D] = 1$. Hence, p-stability implies that $D \leqslant O_p(G \bmod C_G(T)) = O_p(G \bmod Z(T)) = T$. By Lemma 1(b), it follows that $D \leqslant D^*_{(e)}(T) \lhd G$. Hence, setting $W = \langle D^G \rangle$, we have $W \leqslant D^*_{(e)}(T)$. In particular, W is abelian (resp., elementary abelian).

We complete the proof by showing that W satisfies condition $(*)$, whence $W \leqslant D$ and so $D = W \lhd G$, as claimed. By Lemma 2, it will suffice to show that if $x \in S$ with $[W, x, x] = 1$, then $[W, x] = 1$. Choose such an x and let $g \in G$ and $C = C_G(W) \lhd G$. Then p-stability implies that $x \in C_1 = O_p(G \bmod C)$. As $C_1 \lhd G$, $S^g \cap C_1 \in Syl_p(C_1)$. Hence, $C_1 = (S^g \cap C_1)C$. Write $x = cx_1$ with $c \in C$ and $x_1 \in S^g \cap C_1$. As $[W, x, x] = 1$, also $[W, x_1, x_1] = 1$. In particular, $[D^g, x_1, x_1] = 1$. Then, as $x_1 \in S^g$ and $D^g \in \mathcal{D}_{(e)}(S^g)$, we have $[D^g, x_1] = 1$. But then as $x = cx_1$, $[D^g, x] = 1$ for all $g \in G$, whence $[W, x] = 1$, as claimed, which completes the proof. $\square$

Remark. In the context of our theorem, i.e., when $C_G(O_p(G)) \leqslant O_p(G)$, we only require the p-stability condition of [HB, p. 492] for normal p-subgroups P of G.

Next, we provide the promised proof that $D^*(S) \leqslant ZJ(S)$. The proof relies on a replacement result of J. D. Gillam, as refined by Mann [M, Proposition 8]. We include a proof of Gillam's theorem, and an elementary extension, for completeness.

Proposition 1. *Let S be a finite metabelian p-group, and let A be an abelian subgroup of S.*

(a) *If A is an abelian subgroup of S of maximal order, then $\langle A^S \rangle$ contains a normal abelian subgroup B of S with $|B| = |A|$.*
(b) *If A is an elementary abelian subgroup of S of maximal order and S' is elementary abelian as well, then $\langle A^S \rangle$ contains a normal elementary abelian subgroup B of S with $|B| = |A|$.*

Proof. As [M, Proposition 8] gives the proof of (a), we shall give the (almost identical) proof of (b) in detail, and then indicate the minor changes needed for the proof of (a). Let B be an elementary abelian subgroup of maximal order in $\langle A^S \rangle$, chosen so that $|B \cap S'|$ is as large as possible. We claim that B is the desired normal subgroup. Suppose on the contrary that $N_S(B) \neq S$. Choose $x \in N_S(N_S(B)) - N_S(B)$. Then B and B^x normalize each other and are distinct. So $C = BB^x = B[B, x]$ is of class 2 with $B \cap B^x \leqslant Z(C)$.

Suppose $c \in Z(C) - (B \cap B^x)$. We may assume that $c \notin B$. Write $c = bb_1$ with $b \in B$, $b_1 \in B^x$. Then $b_1 \in C_{B^x}(B)$ and so $\langle B, b_1 \rangle$ is elementary abelian of order greater than $|B|$, a contradiction. Hence $Z(C) = B \cap B^x$.

Note that $[B, x] \nleqslant B$. Set $E := Z(C)(B \cap S')[B, x]$. Then

$$E \leqslant C \leqslant \langle B^S \rangle \leqslant \langle A^S \rangle.$$

Since S' is elementary abelian and $E \leqslant Z(C)(C \cap S')$, E is elementary abelian. Also $C = BE$ and $B \cap E \geqslant Z(C) = B \cap B^x$. So

$$\left|E : B \cap B^x\right| \geqslant |E : E \cap B| = |C : B| = \left|BB^x : B\right| = \left|B^x : B \cap B^x\right|.$$

Therefore, $|E| \geqslant |B^x| = |B|$, whence equality holds. However, $[B, x] \leqslant E \cap S'$ and $[B, x] \nleqslant B$, whence $E \cap S' > B \cap S'$, contrary to the choice of B, a contradiction proving (b).

For the proof of (a), it suffices to delete the word elementary from the above proof each of the four times it occurs. $\square$

Now let $\mathcal{A}_e(S)$ denote the set of elementary abelian subgroups of S of maximum order, and let $J_e(S) = \langle A\colon\ A \in \mathcal{A}_e(S)\rangle$. We will use a similar (e) convention as for the D^*-subgroups.

Theorem 2. *Let $D = D^*_{(e)}(S)$. Then*:

(a) *If $D \leqslant H \leqslant S$ and H has class at most 2, then $D \leqslant Z(H)$*;
(b) *D centralizes $Z_2(S)$ and all normal abelian subgroups of S*;
(c) *Every maximal normal (elementary) abelian subgroup of S contains D*; *and*
(d) *$Z(S) \leqslant D^*(S) \leqslant ZJ(S)$ and $\Omega_1(Z(S)) \leqslant D^*_e(S) \leqslant \Omega_1(ZJ_e(S))$.*

Proof. The first statement is clear from the definition. If $x \in Z_2(S)$, then $[D, x, x] = 1$, whence $[D, x] = 1$. Likewise, if A is a normal abelian subgroup of S, then $[D, A, A] \leqslant [A, A] = 1$, and so $[D, A] = 1$ and DA is a normal (elementary) abelian subgroup of S, which proves (b) and (c).

Finally, let $A \in \mathcal{A}_{(e)}(S)$. To prove (d), it will suffice to show that $D \leqslant A$. Set $Q = DA$. Then Q is metabelian (and Q' is elementary abelian if D is). So by Proposition 1, there exists $B \in \mathcal{A}_{(e)}(\langle A^Q\rangle)$ with $B \lhd Q$. It follows from (c) that $D \leqslant B$, and so $Q = BA \leqslant \langle A^Q\rangle$. As Q is nilpotent, it follows that $A = Q$, whence $D \leqslant A$, as claimed. $\square$

We remark that, of course, $D^*_e(S) \leqslant D^*(S)$, by definition. Hence $D^*_e(S) \leqslant \Omega_1(ZJ(S))$, as well.

In conclusion, we describe here the promised family of examples demonstrating that the analogue of Lemma 1(b) does not hold for $ZJ(S)$ in place of $D^*_{(e)}(S)$:

Let p be a prime. Let $G^* = G\langle\sigma\rangle$, where $G = SL(3, p^p)$, the 3-dimensional unimodular matrix group defined over a field F of cardinality p^p, and σ is a field automorphism of order p induced by a Galois automorphism of F. Let $S^* = U\langle\sigma\rangle$, where U is the group of strictly upper triangular matrices in G. Then, S^* is a Sylow p-subgroup of G, and $J(S^*) = U$ is generated by root subgroups of order p^p isomorphic to the additive group of F, on each of which σ acts as a Galois automorphism. As $Z(U)$ is one of these root subgroups, it follows that $Z^* = Z(U)$ is a free σ-module, containing a unique σ-invariant subgroup Z of order p^{p-2}. (In particular, if $p = 2$, then $Z = 1$.)

Let $S = S^*/Z$. If X is either of the two non-central root subgroups of U, then for each non-identity $x \in X$, the map $[\cdot, x] : U \to Z^*$ is a surjective homomorphism with kernel $Z^* \times X$. Let $J = U/Z$. It follows that, for any $z \in Z - \{1\}$, $J/\langle z\rangle$ is an extraspecial p-group, and hence, if A is an abelian subgroup of $J/\langle z\rangle$, then $|A| \leqslant p^{p+1}$. Thus, an abelian subgroup of J has cardinality at most p^{p+2}. Since Z^*X/Z is abelian of cardinality p^{p+2}, $Z^*X/Z \in \mathcal{A}(J)$. As $|C_S(\sigma)| = p^4$, and $C_S(\sigma)$ is not abelian if $p = 2$ (and $S^* = S$), it follows that $J(S) = J$ and $ZJ(S) = Z(J)$ is elementary of order p^2.

Let $T = ZJ(S)\langle\sigma\rangle$. Then T is a non-abelian p-group of order p^3 and exponent p, if p is odd; while $T \cong D_8$, if $p = 2$. In either case, $T = J(T)$. Thus,

$$ZJ(T) = Z(T) < ZJ(S) < T < S.$$

This shows, for all primes p, that the analogue of Lemma 1(b) does not hold with $ZJ(S)$ in place of $D^*_{(e)}(S)$.

References

[GG] G. Glauberman, A characteristic subgroup of a p-stable group, Canad. J. Math. 20 (1968) 1101–1135.
[HB] B. Huppert, N. Blackburn, Finite Groups II, Springer-Verlag, Berlin, 1982.
[Is] I.M. Isaacs, Groups with many equal classes, Duke Math. J. 37 (1970) 501–506.
[M] A. Mann, Groups with few class sizes and the centraliser equality subgroup, Israel J. Math. 142 (2004) 367–380.
[St] B. Stellmacher, A characteristic subgroup of Σ_4-free groups, Israel J. Math. 94 (1996) 367–379.
[W] H. Walter, The characterization of finite groups with abelian Sylow 2-subgroups, Ann. Math. 89 (1969) 405–514.